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Non-Relativistic Derivation of the Non-Euclidean Nature of Space

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The non-Euclidean nature of space (relative to a rotating observer) is derived non-relativistically. Only the law of conservation of energy, Planck's formula, and the equivalence principle are used in the derivation.

KEY WORDS: non-Euclidean space; rotating observers; Planck's formula.

1. INTRODUCTION

In 1909 Paul Ehrenfest published a short paper, entitled *Uniform Rotation of Rigid Bodies and the Theory of Relativity*, in which he pointed out "a contradiction" (Ehrenfast, 1909):

> Consider a relativistically rigid cylinder with radius R and height H. It is given a rotation motion around its axis, which finally becomes constant. As measured by an observer at rest, the radius of the rotating cylinder is R'. The R' has to fulfill the following two contradictory requirements:

1. The circumference of the cylinder must obtain a contraction

$$2\pi R' < 2\pi R \tag{1}$$

relative to its rest length since each of its elements moves with an instantaneous velocity $R'\omega$.

2. If one considers each element along a radius, then the instantaneous velocity of each element is directed perpendicular to the radius. Hence the elements of a radius cannot show any contraction relative to their rest length. This means that:

$$R' = R. \tag{2}$$

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Einstein used the a similar argument to conclude not that there is a contradiction, but that space (relative to a co-rotating observer) is non-Euclidean (Einstein, 1961).

Ehrenfast's "contradiction" as well as Einstein's argument rely on Special Relativity. The question arises, then, whether Special Relativity is a necessary precursor of the discovery of the non-Euclidean nature of space (relative to some observers).

In the present paper we show that Special Relativity is not a necessary precursor by deriving the non-Euclidean character of space (relative to rotating observers) from the conservation of energy, Planck's formula $E = hf$, and the equivalence principle alone.

The derivation itself is given in Section 2. Since we use the formula $E = mc^2$ in the derivation, it would seem as if Special Relativity is invoked. In Section 3 we point out that this formula is used only for particles at rest and prove its validity for such particles non-relativistically. Finally, in Section 4 we discuss an issue that has to do with the very natural assumption that if the velocity of observer A relative to observer B is v, then the velocity of observer B relative to observer A is $-v$.

2. THE DERIVATION

In his *Lectures on Physics*, Vol. 2, Chap. 42, Feynman presents three different proofs of the dependence of the rate of clocks on their position in a gravitational field (Feynman *et al.*, 1977). The second and third derivations are based on the conservation of energy. Our derivation is inspired by the second of these proofs.

Consider an atom that can be in two energy states, a ground state E_0 and an excited state E_1. Using units with $c = 1$, $E = mc^2$ becomes $E = m$ (i.e., these energies are equal to the masses of the atom in its ground and excited states).

The derivation now proceeds as follows. Suppose we have such an atom in state E_1 at a distance R from the center of a rotating disc. Next, we carry it and place it at the center. The amount of work needed for that is $\frac{1}{2} E_1\omega^2 R^2$, where ω is the angular velocity. Now we let the atom emit a photon and go into the ground state E_0. We then carry the atom back to its original place. On the return trip the mass is E_0; we get back the energy $\frac{1}{2} E_0\omega^2 R^2$. Altogether the amount of work we did is

$$\Delta U = (E_1 - E_0)\frac{1}{2}\,\omega^2 R^2 \tag{3}$$

When the atom emitted a photon, it gave up the energy $E_1 - E_0$.

If the photon happened to go to the rim, a distance R from the center, and be absorbed, the amount of energy it would deliver there is calculated as follows: We started with energy E_1 at the rim. When we finish, the energy at the rim is the energy E_0 of the atom in its lower (ground) state plus the energy E_{ph} received from the photon. Since we supplied the additional energy ΔU, conservation of energy demands

$$E_{\text{ph}} + E_0 = E_1 + \Delta U \tag{4}$$

i.e.,

$$E_{\text{ph}} = E_1 - E_0 + \Delta U \tag{5}$$

or

$$E_{\text{ph}} = (E_1 - E_0)\left[1 + \tfrac{1}{2}\,\omega^2 R^2\right] \tag{6}$$

But a photon with energy E_{ph} has frequency $f = E_{\text{ph}}/h$. Remembering that

$$f_0 = \frac{E_1 - E_0}{h} \tag{7}$$

is the frequency of the emitted photon, we conclude that

$$f = f_0\left[1 + \tfrac{1}{2}\,\omega^2 R^2\right]. \tag{8}$$

The factor in the square brackets represents the factor by which a clock at the rim clicks faster than a clock at the center.

Consider now an inertial observer A at a distance R from the center of rotation and at rest relative to the center, and another observer, B, at a distance R from the center, rotating at speed $v = \omega R$. Since observer A is at rest relative to the center, then both relativistically and non-relativistically his clock moves at the same rate as the clock at the center. Now let T and T' be the times it takes for one revolution according to observers A and B respectively. According to Equation (8), $T \neq T'$; hence the circumference of the circle of radius R, which is vT and vT' respectively (v is the relative speed of the two observers) is also different. According to observer A the circumference is $2\pi R$, but since $T \neq T'$, according to observer B the circumference is not $2\pi R$. The space of observer B is non-Euclidean.

3. A NON-RELATIVISTIC DERIVATION OF THE FORMULA $E = mc^2$ FOR PARTICLES AT REST

In the above derivation we have used the formula $E = mc^2$. It seems as if by doing that, we have used Special Relativity. However, this is not the case. Notice that the formula is used only to calculate the energy difference between the excited and the ground state of the atom when it is at rest. For such atoms the proof of the formula does not use Special Relativity.

Consider Equation (5):

$$E_{\text{ph}} = E_1 - E_0 + \Delta U$$

If an atom that is lifted from the floor to a distance H above it, where a photon is emitted, then

$$\Delta U = (m_1 - m_0)\, gH \tag{9}$$

Consider now the first of Feynman's three proofs. It involves applying the equivalence principle (which is based on the equality of the inertial and gravitational masses in Newton's law of gravitation) to the Doppler effect, which is purely kinematical. Furthermore, Feynman is using the non-relativistic Doppler formula, since terms that involve v^2 and high powers of v are ignored. He then obtains

$$f = f_0(1 + gH) \tag{10}$$

This agrees with Equations (5) and (9)

$$E_{\text{ph}} = E_1 - E_0 + \Delta U = E_1 - E_0 + (m_1 - m_0)\, gH \tag{11}$$

if and only if

$$E_1 - E_0 = m_1 - m_0 \tag{12}$$

Since this equation holds for arbitrary values of the two energies and two masses (and since $c = 1$),

$$E = mc^2 + A \tag{13}$$

where A is an arbitrary (i.e., immeasurable) constant, the most convenient choice for which is $A = 0$. The choice of A does not really matter, because Equation (12) is used to calculate the energy difference $E_1 - E_0$.

4. THE ISSUE OF RELATIVE VELOCITIES

A final consideration: Let v be the velocity of the observer at A as seen by the observer B, and v'—the velocity of observer B as seen by observer A.

Naturally, we assumed in our derivation that $v' = -v$. We will prove now that our derivation holds even if $v' \neq -v$: To spoil our proof, the value of v' must be such that the circumference of the circle according to observer B is $2\pi R$, same as according to observer A. For this we must have

$$v'/v = T/T' \tag{14}$$

T/T' depends only on $v = \omega R$, not on ω and R separately. Therefore we can make R arbitrarily large and ω arbitrarily small, keeping v'/v the same. If R is big enough, the acceleration $\omega^2 R$ becomes arbitrarily small, and the movement of observer B approaches a movement on a straight line. For such a movement, by Galileo (or Descartes') principle of relativity, it is true by symmetry that $v' = -v$.

Hence if there is departure from the relation $v' = -v$, it would be gradual, with $v' - (-v) = v' + v$ increasing as R decreases (and ω increases to keep $v = \omega R$ the same); v' cannot have the fixed value demanded by Equation (14).

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